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Stability in Nonlinear Neutral Differential Equations with Infinite Delay

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Abstract. In this paper we use the contraction mapping theorem to obtain asymptotic stability results of the nonlinear neutral differential equation with infinite delay

$$\frac{\mathrm{d}}{\mathrm{d}\,t}x(t) = -a(t)x(t-\tau_1(t)) + \frac{\mathrm{d}}{\mathrm{d}\,t}Q(t, x(t-\tau_2(t))) + \int_{-\infty}^{t} D(t,s)f(x(s))\,\mathrm{d}\,s.$$

An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some results due to Burton [6], Zhang [17], Althubiti, Makhzoum, Raffoul [1].

1. Introduction

Certainly, the Lyapunov direct method has been successfully used to investigate stability properties of a wide variety of ordinary, functional and partial differential equations. Nevertheless, the application of this method to problems of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms [4–6]. Recently, investigators such as Burton, Zhang, Raffoul and others have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory see ([1]-[15], [17]). The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [4]).

In this paper we consider the nonlinear neutral differential equation with infinite delay

$$\frac{\mathrm{d}}{\mathrm{d}\,t}x(t) = -a(t)x(t-\tau_1(t)) + \frac{\mathrm{d}}{\mathrm{d}\,t}Q(t, x(t-\tau_2(t))) + \int_{-\infty}^{t} D(t,s)f(x(s))\,\mathrm{d}\,s, \tag{1}$$

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with the initial condition

$$x(t) = \psi(t) \text{ for } t \in (-\infty, t_0],$$

where $\psi \in C((-\infty, t_0], \mathbb{R})$ is bounded. Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 arrow S_2$ with the supremum norm $\|.\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+, \mathbb{R})$, $D \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ and $\tau_1, \tau_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_1(t) \to \infty$ and $t - \tau_2(t) \to \infty$ as $t \to \infty$. The functions $Q(t, x)$ and $f(x)$ are globally Lipschitz continuous in x. That is, there are positive constants L_1 and L_2 such that

$$|Q(t,x) - Q(t,y)| \le L_1 \|x - y\|, \tag{2}$$

and

$$|f(x) - f(y)| \le L_2 \|x - y\|. \tag{3}$$

Also, there is positive constant L_3 such that

$$\int_{-\infty}^{t} |D(t,s)| \,\mathrm{d}\, s \le L_3. \tag{4}$$

So, we assume that

$$Q(t, 0) = f(0) = 0. \tag{5}$$

Equation (1) and its special cases have been investigated by many authors. For example, Burton in [6], and Zhang in [17] have studied the equation

$$x'(t) = -a(t)x(t - \tau_1(t)), \tag{6}$$

and proved the following.

Theorem A (Burton [6]). *Suppose that $\tau_1(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$\begin{aligned} &\int_{t-r}^{t} |a(s+r)| \,\mathrm{d}\, s \\ &\quad + \int_0^t |a(s+r)| e^{-\int_s^t a(u+r)\,\mathrm{d}\, u} \left(\int_{s-r}^{s} |a(u+r)| \,\mathrm{d}\, u \right) \mathrm{d}\, s \le \alpha, \end{aligned} \tag{7}$$

for all $t \ge 0$ and $\int_0^\infty a(s)ds = \infty$. Then, for every continuous initial function $\psi : [-r, 0] \to \mathbb{R}$, the solution $x(t) = x(t, 0, \psi)$ of (6) is bounded and tends to zero as $\to \infty$.

Theorem B (Zhang [17]). *Suppose that τ_1 is differentiable, the inverse function g of $t - \tau_1(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for $t \ge 0$, $\liminf\limits_{t\to\infty} \int_0^t a(g(s)) \,\mathrm{d}\, s > -\infty$ and*

$$\begin{aligned} &\int_{t-\tau_1(t)}^{t} |a(g(s))| \,\mathrm{d}\, s + \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\, u} |a(s)||\tau_1'(s)| \,\mathrm{d}\, s \\ &\quad + \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\, u} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(u))| \,\mathrm{d}\, u \right) \mathrm{d}\, s \le \alpha. \end{aligned} \tag{8}$$

Then the zero solution of (6) is asymptotically stable if and only if $\int_0^t a(g(s))\,\mathrm{d}\,s \to \infty$ *as* $t \to \infty$.

Obviously, Theorem B improves Theorem A. On the other hand, Althubiti, Makhzoum, Raffoul in [1] considered the following nonlinear neutral differential equation

$$(9)\quad \frac{\mathrm{d}}{\mathrm{d}\,t}x(t) = -a(t)x(t) + \frac{\mathrm{d}}{\mathrm{d}\,t}Q(t, x(t-\tau_2(t))) + \int_{-\infty}^{t} D(t,s)f(x(s))\,\mathrm{d}\,s,$$

and obtained the following.

Theorem C (Althubiti, Makhzoum, Raffoul [1]). *Suppose (2)–(5) hold, and there exists a constant* $\alpha \in (0,1)$ *such that for* $t \geq 0$, $\int_0^t a(s)\,\mathrm{d}\,s \to \infty$ *as* $t \to \infty$, *and*

$$(10)\qquad L_1 + \int_0^t e^{-\int_s^t a(u)\,\mathrm{d}\,u}[L_1|a(s)| + L_2L_3]\,\mathrm{d}\,s \leq \alpha.$$

Then every solution $x(t) = x(t, 0, \psi)$ *of (9) with a small continuous initial function* ψ *is bounded and tends to zero as* $t \to \infty$.

Our purpose here is to give, by using the contraction mapping principle, asymptotic stability results of a nonlinear neutral differential equation with infinite delay (1). An asymptotic stability theorem with a necessary and sufficient condition is proved. The results presented in this paper improve and generalize the main results in [1, 6, 17].

2. Main results

For each $(t_0, \psi) \in \mathbb{R}^+ \times C((-\infty, t_0], \mathbb{R})$, a solution of (1) through (t_0, ψ) is a continuous function $x : (-\infty, t_0+\sigma) \to \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1) on $[t_0, t_0+\sigma)$ and $x = \psi$ on $(-\infty, t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. For each $(t_0, \psi) \in \mathbb{R}^+ \times C((-\infty, t_0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1) defined on $[t_0, \infty)$. For fixed t_0, we define $\|\psi\| = \sup\{|\psi(t)| : -\infty < t \leq t_0\}$. Stability definitions may be found in [4], for example.

Our aim here is to generalize Theorem B and Theorem C to (1).

Theorem 2.1. *Suppose (2)–(5) hold. Let* τ_j *be differentiable, and suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0,1)$ *such that for* $t \geq 0$

$$(11)\qquad \liminf_{t\to\infty} \int_0^t H(s)\,\mathrm{d}\,s > -\infty,$$

and

$$
\begin{aligned}
&L_1+\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|h_j(s)|\,\mathrm{d}\,s\\
&+\int_0^t e^{-\int_s^t H(u)\,\mathrm{d}\,u}\{|-a(s)+h_1(s-\tau_1(s))(1-\tau_1'(s))|\\
&+|h_2(s-\tau_2(s))(1-\tau_2'(s))|+L_1|H(s)|+L_2L_3\}\,\mathrm{d}\,s\\
&+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)\,\mathrm{d}\,u}|H(s)|\left(\int_{s-\tau_j(s)}^{s}|h_j(u)|\,\mathrm{d}\,u\right)\mathrm{d}\,s\le\alpha,
\end{aligned}
\tag{12}
$$

where $H(t)=\sum_{j=1}^{2}h_j(t)$. *Then the zero solution of (1) is asymptotically stable if and only if*

$$
\int_0^t H(s)\,\mathrm{d}\,s\to\infty \quad \text{as} \quad t\to\infty. \tag{13}
$$

Proof. First, suppose that (13) holds. For each $t_0\ge 0$, we set

$$
K=\sup_{t\ge 0}\{e^{-\int_0^t H(s)\,\mathrm{d}\,s}\}. \tag{14}
$$

Let $\psi\in C((-\infty,t_0],\mathbb{R})$ be fixed and define

$$
S=\{\varphi\in C(\mathbb{R},\mathbb{R}):\varphi(t)\to 0 \text{ as } t\to\infty,\quad \varphi(t)=\psi(t) \text{ for } t\in(-\infty,t_0]\}.
$$

This S is a complete metric space with metric $\rho(x,y)=\sup_{t\ge t_0}\{|x(t)-y(t)|\}$.

Multiply both sides of (1) by $e^{\int_{t_0}^t H(u)\,\mathrm{d}\,u}$ and then integrate from t_0 to t to obtain

$$
\begin{aligned}
x(t)&=(\psi(t_0)-Q(t_0,\psi(t_0-\tau_2(t_0))))e^{-\int_{t_0}^t H(u)\,\mathrm{d}\,u}\\
&+Q(t,x(t-\tau_2(t)))+\sum_{j=1}^{2}\int_{t_0}^{t}e^{-\int_s^t H(u)\,\mathrm{d}\,u}h_j(s)x(s)\,\mathrm{d}\,s\\
&+\int_{t_0}^{t}e^{-\int_s^t H(u)\,\mathrm{d}\,u}\Big\{-a(s)x(s-\tau_1(s))-H(s)Q(s,x(s-\tau_2(s)))\\
&+\int_{-\infty}^{s}D(s,u)f(x(u))\,\mathrm{d}\,u\Big\}\,\mathrm{d}\,s.
\end{aligned}
$$

Performing an integration by parts, we have

$$\begin{aligned}
x(t) &= \big(\psi(t_0) - Q\big(t_0, \psi(t_0 - \tau_2(t_0))\big)\big)e^{-\int_{t_0}^{t} H(u)\,\mathrm{d}\,u} \\
&+ Q(t, x(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}\,\mathrm{d}\left(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,\mathrm{d}\,u\right) \\
&+ \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}\{h_j(s-\tau_j(s))(1-\tau_j'(s))\}x(s-\tau_j(s))\,\mathrm{d}\,s \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}\Big\{-a(s)x(s-\tau_1(s)) - H(s)Q(s, x(s-\tau_2(s))) \\
&+ \int_{-\infty}^{s} D(s,u)f(x(u))\,\mathrm{d}\,u\Big\}\,\mathrm{d}\,s.
\end{aligned}$$

Thus,

$$\begin{aligned}
x(t) &= \big(\psi(t_0) - Q(t_0, \psi(t_0 - \tau_2(t_0))) \\
&- \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,\mathrm{d}\,s\big) \times e^{-\int_{t_0}^{t} H(u)\,\mathrm{d}\,u} \\
&+ Q(t, x(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} h_j(s)x(s)\,\mathrm{d}\,s \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}\Big\{(-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s)))x(s-\tau_1(s)) \\
&+ h_2(s-\tau_2(s))(1-\tau_2'(s))x(s-\tau_2(s))\Big\}\,\mathrm{d}\,s \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}\Big\{-H(s)Q(s, x(s-\tau_2(s))) \\
&+ \int_{-\infty}^{s} D(s,u)f(x(u))\,\mathrm{d}\,u\Big\}\,\mathrm{d}\,s \\
&- \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)\,\mathrm{d}\,u}H(s)\left(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,\mathrm{d}\,u\right)\mathrm{d}\,s.
\end{aligned} \tag{15}$$

Use (15) to define the operator $P : S \to S$ by $(P\varphi)(t) = \psi(t)$ for $t \in (-\infty, t_0]$ and

$$
\begin{aligned}
(P\varphi)(t) &= \Big\{\psi(t_0) - Q(t_0, \psi(t_0 - \tau_2(t_0))) \\
&- \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,\mathrm{d}\,s\Big\} \times e^{-\int_{t_0}^{t} H(u)\,\mathrm{d}\,u} \\
&+ Q(t, \varphi(t - \tau_2(t))) + \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} h_j(s)\varphi(s)\,\mathrm{d}\,s \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u}\Big\{(-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s)))\varphi(s - \tau_1(s)) \\
&+ h_2(s - \tau_2(s))(1 - \tau_2'(s))\varphi(s - \tau_2(s))\Big\}\,\mathrm{d}\,s \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u}\Big\{-H(s)Q(s, \varphi(s - \tau_2(s))) \\
&+ \int_{-\infty}^{s} D(s,u) f(\varphi(u))\,\mathrm{d}\,u\Big\}\,\mathrm{d}\,s \\
&- \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u)\varphi(u)\,\mathrm{d}\,u\right)\mathrm{d}\,s,
\end{aligned} \tag{16}
$$

for $t \geq t_0$. It is clear that $(P\varphi) \in C(\mathbb{R}, \mathbb{R})$. We now show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > t_0$ such that $s \geq T_1$ implies that $|\varphi(s - \tau_j(s))| < \varepsilon$ for $j = 1, 2$. Thus, for $t \geq T_1$, the last term I_6 in (16) satisfies

$$
\begin{aligned}
|I_6| &= \Big|\sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u)\varphi(u)\,\mathrm{d}\,u\right)\mathrm{d}\,s\Big| \\
&\leq \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)||\varphi(u)|\,\mathrm{d}\,u\right)\mathrm{d}\,s \\
&+ \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)||\varphi(u)|\,\mathrm{d}\,u\right)\mathrm{d}\,s \\
&\leq \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,\mathrm{d}\,u\right)\mathrm{d}\,s \\
&+ \varepsilon \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,\mathrm{d}\,u\right)\mathrm{d}\,s.
\end{aligned}
$$

By (13), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,\mathrm{d}\,u \right) \mathrm{d}\,s < \varepsilon.$$

Apply (12) to obtain $|I_6| < \varepsilon + \alpha\epsilon < 2\varepsilon$. Thus, $I_6 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (16) approach zero as $t \to \infty$. This yields $(P\varphi)(t) \to 0$ as $t \to \infty$, and hence $P\varphi \in S$. Also, by (12), P is a contraction mapping with contraction constant α. By the contraction mapping principle (Smart [16], p. 2), P has a unique fixed point x in S which is a solution of (1) with $x(t) = \psi(t)$ on $(-\infty, t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying $2\delta K e^{\int_0^{t_0} H(u)\,\mathrm{d}\,u} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of (1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (16). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $(-\infty, t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $-\infty < s < t^*$, then it follows from (16) that

$$\begin{aligned}
|x(t^*)| &\leq \|\psi\| \Big(1 + L_1 + \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} |h_j(s)|\,\mathrm{d}\,s \Big) e^{-\int_{t_0}^{t^*} H(u)\,\mathrm{d}\,u} \\
&\quad + L_1\epsilon + \epsilon \sum_{j=1}^{2} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s)|\,\mathrm{d}\,s \\
&\quad + \epsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)\,\mathrm{d}\,u} \Big\{ |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| \\
&\quad + |h_2(s-\tau_2(s))(1-\tau_2'(s))| + L_1|H(s)| + L_2L_3 \Big\} \mathrm{d}\,s \\
&\quad + \epsilon \sum_{j=1}^{2} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)\,\mathrm{d}\,u} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,\mathrm{d}\,u \right) \mathrm{d}\,s \\
&\leq 2\delta K e^{\int_0^{t_0} H(u)\,\mathrm{d}\,u} + \alpha\varepsilon < \epsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, $|x(t)| < \varepsilon$ for all $t \geq t_0$, and the zero solution of (1) is stable. This shows that the zero solution of (1) is asymptotically stable if (13) holds.

Conversely, suppose (13) fails. Then by (11) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(u)\,\mathrm{d}\,u = l$ for some $l \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} H(u)\,\mathrm{d}\,u \leq J,$$

for all $n \geq 1$. To simplify our expressions, we define

$$\begin{aligned}\omega(s) &= |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| + L_2L_3 \\ &\quad + |h_2(s-\tau_2(s))(1-\tau_2'(s))| + |H(s)|\Big(L_1 + \sum_{j=1}^{2}\int_{s-\tau_j(s)}^{s}|h_j(u)|\,\mathrm{d}\,u\Big),\end{aligned}$$

for all $s \geq 0$. By (12), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)\,\mathrm{d}\,u}\omega(s)\,\mathrm{d}\,s \leq \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\,u}\omega(s)\,\mathrm{d}\,s \leq \alpha e^{\int_0^{t_n} H(u)\,\mathrm{d}\,u} \leq e^J.$$

The sequence $\{\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\,u}\omega(s)\,\mathrm{d}\,s = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\,u}\omega(s)\,\mathrm{d}\,s < \delta_0/4K,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \leq 1$.

By *(11)*, K in *(14)* is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of *(1)* with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\psi(t_m) - Q(t_m, \psi(t_m - \tau_2(t_m))) - \sum_{j=1}^{2}\int_{t_m-\tau_j(t_m)}^{t_m} h_j(s)\psi(s)\,\mathrm{d}\,s \geq \frac{1}{2}\delta_0.$$

It follows from (16) with $x(t) = (Px)(t)$ that for $n \geq m$

$$
\begin{aligned}
&|x(t_n) - Q(t_n, x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n - \tau_j(t_n)}^{t_n} h_j(s) x(s) \,\mathrm{d}\, s| \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,\mathrm{d}\, u} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)\,\mathrm{d}\, u} \omega(s) \,\mathrm{d}\, s \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,\mathrm{d}\, u} - e^{-\int_0^{t_n} H(u)\,\mathrm{d}\, u} \int_{t_m}^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\, u} \omega(s) \,\mathrm{d}\, s \\
&= e^{-\int_{t_m}^{t_n} H(u)\,\mathrm{d}\, u} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)\,\mathrm{d}\, u} \int_{t_m}^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\, u} \omega(s) \,\mathrm{d}\, s \right) \\
&\geq e^{-\int_{t_m}^{t_n} H(u)\,\mathrm{d}\, u} \left(\frac{1}{2}\delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s H(u)\,\mathrm{d}\, u} \omega(s) \,\mathrm{d}\, s \right) \\
&\geq \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,\mathrm{d}\, u} \geq \frac{1}{4}\delta_0 e^{-2J} > 0.
\end{aligned} \tag{17}
$$

On the other hand, if the zero solution of (1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (12) holds, we have

$$
x(t_n) - Q(t_n, x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n - \tau_j(t_n)}^{t_n} h_j(s) x(s) \,\mathrm{d}\, s \to 0 \text{ as } n \to \infty,
$$

which contradicts (17). Hence condition (13) is necessary for the asymptotic stability of the zero solution of (1) The proof is complete. □

Remark 2.1. It follows from the first part of the proof of Theorem 2.1 that the zero solution of (1) is stable under (11) and (12). Moreover, Theorem 2.1 still holds if (12) is satisfied for $t \geq t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

For the special case $Q(t,x) = 0$ and $D(t,s) = 0$, we can get

Corollary 2.1. *Let τ_1 be differentiable, and suppose that there exist continuous function $h_1 : [m_1(t_0), \infty) \to \mathbb{R}$ where $m_1(t_0) = \inf\{t - \tau_1(t),\ t \geq t_0\}$, and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$
\liminf_{t \to \infty} \int_0^t h_1(s) \,\mathrm{d}\, s > -\infty,
$$

and

$$
\begin{aligned}
&\int_{t-\tau_1(t)}^{t} |h_1(s)| \,\mathrm{d}\, s \\
&+ \int_0^t e^{-\int_s^t h_1(u)\,\mathrm{d}\, u} |-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))| \,\mathrm{d}\, s \\
&+ \int_0^t e^{-\int_s^t h_1(u)\,\mathrm{d}\, u} |h_1(s)| \left(\int_{s-\tau_1(s)}^{s} |h_1(u)| \,\mathrm{d}\, u \right) \mathrm{d}\, s \leq \alpha.
\end{aligned} \tag{18}
$$

Then the zero solution of (6) is asymptotically stable if and only if

$$\int_0^t h_1(s)\,\mathrm{d}\,s \to \infty \quad as \quad t \to \infty.$$

Remark 2.2. When $\tau_1(s) = r$, a constant, $h_1(s) = a(s+r)$, Corollary 2.1 reduces to Theorem A. When $h_1(s) = a(g(s))$, where $g(s)$ is the inverse function of $s - \tau_1(s)$, Corollary 2.1 reduces to Theorem B.

We give an example to illustrate the applications of Corollary 2.1.

Example 2.1. Consider the following linear delay differential equation

$$x'(t) = -a(t)x(t - \tau_1(t)), \tag{19}$$

where $\tau_1(t) = 0.285t$, $a(t) = 1/(0.715t + 1)$. Then the zero solution of (19) is asymptotically stable.

Proof. Choosing $h_1(t) = 1.25/(t+1)$ in Corollary 2.1, we have

$$\begin{aligned}\int_{t-\tau_1(t)}^t |h_1(s)|\,\mathrm{d}\,s &= \int_{0.715t}^t \frac{1.25}{s+1}\,\mathrm{d}\,s \\ &= 1.25 \ln \frac{t+1}{0.715t+1} < 0.4194,\end{aligned}$$

$$\begin{aligned}&\int_0^t e^{-\int_s^t h_1(u)\,\mathrm{d}\,u} |h_1(s)| \left(\int_{s-\tau_1(s)}^s |h_1(u)|\,\mathrm{d}\,u \right) \mathrm{d}\,s \\ &\qquad < \int_0^t e^{-\int_s^t (1.25/(u+1))\,\mathrm{d}\,u} \frac{1.25}{1+s} \times 0.4194\,\mathrm{d}\,s < 0.4194,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t e^{-\int_s^t h_1(u)\,\mathrm{d}\,u} |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))|\,\mathrm{d}\,s \\ &\quad = \int_0^t e^{-\int_s^t (1.25/(u+1))\,\mathrm{d}\,u} \frac{1 - 1.25 \times 0.715}{0.715s+1}\,\mathrm{d}\,s \\ &\quad < \frac{1 - 1.25 \times 0.715}{1.25 \times 0.715} \int_0^t e^{-\int_s^t (1.25/(u+1))\,\mathrm{d}\,u} \frac{1.25}{s+1}\,\mathrm{d}\,s < 0.1189.\end{aligned}$$

It is easy to see that all the conditions of Corollary 2.1 hold for $\alpha = 0.4194 + 0.4194 + 0.1189 = 0.9577 < 1$. Thus, Corollary 2.1 implies that the zero solution of (19) is asymptotically stable.

However, Theorem B cannot be used to verify that the zero solution of (19) is asymptotically stable. In fact, $a(g(t)) = 1/(t+1)$. As $t \to \infty$,

$$\begin{aligned}\int_{t-\tau_1(t)}^t |a(g(s))|\,\mathrm{d}\,s &= \int_{0.715t}^t \frac{1}{s+1}\,\mathrm{d}\,s \\ &= \ln \frac{t+1}{0.715t+1} \to -\ln(0.715),\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(g(s))|\left(\int_{s-\tau_1(s)}^s |a(g(s))|\,\mathrm{d}\,u\right)\mathrm{d}\,s \\
&= \int_0^t e^{-\int_s^t (1/(u+1))\,\mathrm{d}\,u}\frac{1}{1+s}\left(\int_{0.715s}^s \frac{1}{u+1}\,\mathrm{d}\,u\right)\mathrm{d}\,s \\
&= \frac{1}{t+1}\int_0^t [\ln(s+1)-\ln(0.715s+1)]\,\mathrm{d}\,s \to -\ln(0.715),
\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(s)||\tau_1'(s)|\,\mathrm{d}\,s \\
&\qquad = \frac{0.285}{t+1}\int_0^t \frac{s+1}{0.715s+1}\,\mathrm{d}\,s \\
&\qquad = \frac{0.285}{0.715}\frac{t}{t+1} - \left(\frac{0.285}{0.715}\right)^2\frac{\ln(0.715t+1)}{t+1} \to \frac{0.285}{0.715}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\overline{\lim_{t\geq 0}}\Big\{\int_{t-\tau_1(t)}^t |a(g(s))|\,\mathrm{d}\,s + \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(s)||\tau_1'(s)|\,\mathrm{d}\,s \\
&+ \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(g(s))|\Big(\int_{s-\tau_1(s)}^s |a(g(s))|\,\mathrm{d}\,u\Big)\,\mathrm{d}\,s\Big\} \\
&= -2\ln(0.715) + \frac{0.285}{0.715} \simeq 1.0695.
\end{aligned}$$

In addition, the left-hand side of the following inequality is increasing in $t>0$, then there exists some $t_0>0$ such that for $t>t_0$,

$$\begin{aligned}
&\int_{t-\tau_1(t)}^t |a(g(s))|\,\mathrm{d}\,s + \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(s)||\tau_1'(s)|\,\mathrm{d}\,s \\
&+ \int_0^t e^{-\int_s^t a(g(u))\,\mathrm{d}\,u}|a(g(s))|\Big(\int_{s-\tau_1(s)}^s |a(g(s))|\,\mathrm{d}\,u\Big)\,\mathrm{d}\,s > 1.069.
\end{aligned}$$

This implies that condition (8) does not hold. Thus, Theorem B cannot be applied to equation (19). □

Letting $\tau_1 = 0$, we have

Corollary 2.2. *Suppose (2)–(5) hold. Let τ_2 be differentiable, and suppose that there exist continuous functions $h_j : [m_j(t_0),\infty) \to \mathbb{R}$ for $j=1,2$ and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)\,\mathrm{d}\,s > -\infty,$$

and

$$
\begin{aligned}
L_1 &+ \int_{t-\tau_2(t)}^{t} |h_2(s)|\,\mathrm{d}\,s \\
&+ \int_0^t e^{-\int_s^t H(u)\,\mathrm{d}\,u}\Big\{|-a(s)+h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s))| \\
&\qquad\qquad + L_1|H(s)| + L_2L_3\Big\}\,\mathrm{d}\,s \\
&+ \int_0^t e^{-\int_s^t H(u)\,\mathrm{d}\,u}|H(s)|\Big(\int_{s-\tau_2(s)}^{s} |h_2(u)|\,\mathrm{d}\,u\Big)\,\mathrm{d}\,s \le \alpha,
\end{aligned}
\tag{20}
$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$. *Then the zero solution of (9) is asymptotically stable if and only if*

$$\int_0^t H(s)\,\mathrm{d}\,s \to \infty \text{ as } t \to \infty.$$

Remark 2.3. When $h_1(s) = a(s)$ and $h_2(s) = 0$, Corollary 2.2 reduces to Theorem C.

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